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A complete basis set of functionally independent invariants of continuous Lie groups

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Abstract. Using the methods of differential equations we have obtained the complete set of functionally independent bases of invariants for some continuous Lie groups of mathematical physics: (i) the conformal group; (ii) the semi-direct product of the Weyl group and the one-parameter group of Ψ translation; (iii) the direct product of the Poincaré group and the one-parameter group of Ψ scaling; (iv) the Weyl group; (v) the 14-parameter maximal symmetry group of the Schrödinger equation for the Coulombic system of atoms and molecules; and (vi) SU(3) of elementary particle physics. The first four groups are the only groups which, together with the Poincaré group, are the maximal symmetry groups of all forms of the non-linear Klein-Gordon equation.

1. Introduction

In group theoretical studies of physical problems the knowledge of the symmetry group is, of course, the starting point. An important aspect of group theoretical analysis involves knowledge of the invariants of the system. In classical problems there are two methods of attack: Noether's method, based on the invariance of the action integral (Lutzky 1978, 1979), and Lie's theory of extended groups (Leach 1981, Prince 1983, Prince and Eliezer 1980, 1981).

In the investigations of quantum systems the Casimir operators play the same role as the classical invariants. These polynomials of the generators of the symmetry group of a dynamical system are important because the simultaneous eigenvalues of the complete set of algebraically independent Casimir operators characterise the irreducible finite-dimensional representations. Berdjis (1981) has given a criterion for a set of Casimir operators of a finite-dimensional complex Lie group to be complete and has used this criterion to construct the complete set. It is to be noted that this method is used only for polynomial operators.

Winternitz and co-workers (Patera *et al* 1976a, b) have considered the complete set of functionally independent (in contrast to algebraically independent) invariants of Lie groups. Any invariant of a particular Lie group can thus be functionally expressed (not just algebraically expressed) in terms of the members of this complete basis set. Their method consists of giving a representation of the generators of the Lie group in the form of differential operators. Obtaining the complete basis set reduces to obtaining the complete set of integrals of a set of partial differential equations. However, they do not give an algorithm for the complete solution of the set of partial differential equations. We point out here that exactly such an algorithm exists (Goursat 1945) and this procedure has been applied for obtaining the complete basis set of functionally independent invariants.

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The groups that we have investigated are the following. The non-linear Klein-Gordon equation in three space and one time dimensions has, for general non-linearity, the Poincaré group as the maximal symmetry group. The symmetry group contains the Poincaré group as a proper subgroup only for special types of non-linearity (Rudra 1986a). The corresponding groups are (i) the conformal group; (ii) the semi-direct product of the Weyl group and the one-parameter Ψ translation; (iii) the direct product of the Poincaré group and the one-parameter group of Ψ scaling; and (iv) the Weyl group. The next system that we studied is (v) the 14-parameter maximal symmetry group of the Schrödinger equation of the Coulombic system of atoms and molecules (Rudra 1986b). The last case is (vi) the SU(3) group of elementary particle physics.

2. Complete basis set of invariants

In this section we first define what we mean by a complete basis set of functionally independent invariants of a Lie group. We then describe the method for obtaining this complete set for a given group.

We take the algebraic structure of a Lie group G by the commutation relations of its generators X_i, $i = 1, \ldots, n$,

$$[X_i, X_j] = \sum_k C_{ij}^k X_k \qquad i, j, k = 1, \ldots, n. \tag{1}$$

A function $I(\{X\})$ of the generators will be an invariant if

$$[X_i, I] = 0 \qquad \text{for all } i = 1, \ldots, n. \tag{2}$$

The set of invariants I_α, $\alpha = 1, \ldots, p$, is a complete basis set of functionally independent invariants of G if any invariant I of G can be expressed as a function of the I_α and the I_α are functionally independent.

In order to obtain this complete set we start with (Patera *et al* 1976a, b) a representation of the X_i in the form of differential operators:

$$X_i \rightarrow \chi_i = \sum_{kj} C_{ij}^k x_k \partial/\partial x_j. \tag{3}$$

For an invariant $I(\{X\})$ we get a partial differential equation

$$\chi_i I(\{x\}) = 0 \qquad i = 1, \ldots, n \tag{4}$$

comparable to equation (2). All these n equations may not be functionally independent. Let s of them be functionally independent and these s equations form a complete Jacobian system of equations. These n partial differential equations, equation (4), in n variables x_i, $i = 1, \ldots, n$, will have $p = n - s$ integrals, $I_\alpha(\{x\})$, $\alpha = 1, \ldots, p$, such that

$$\chi_i I_\alpha = 0 \qquad i = 1, \ldots, n, \alpha = 1, \ldots, p \tag{5}$$

and any $I(\{x\})$ satisfying equation (4) will be a function of I_α.

We now replace the x_i in I_α by X_i and symmetrise the expressions with respect to the X_i. In case rational functions of x_i occur in I_α we symmetrise the numerator and denominator separately. The resulting functions

$$I_\alpha(\{x\}) \rightarrow I_\alpha(\{X\}) \tag{6}$$

form the complete set of functionally independent bases of invariants of the Lie group.

We now describe the method (Goursat 1945) for obtaining the $p = n - s$ independent integrals of the set of equations (4). We write the s Jacobian equations in the form

$$L_i I \equiv \partial I/\partial x_i + \sum_{m=s+1}^{n} b_{im}(x)\partial I/\partial x_m = 0 \qquad i = 1, \ldots, s. \tag{7}$$

We take any one of them, say the first one, $L_1 I = 0$, and obtain the integrals $x_i^{(1)}$, $i = 2, \ldots, n$. We then choose variables $x_1^{(1)}$ and $x_i^{(1)}$, $i = 2, \ldots, n$, so that the Jacobian determinant $\partial(\{x^{(1)}\})/\partial(\{x\}) \neq 0$. In terms of these variables we rewrite equation (7) and obtain

$$L_1^{(1)} I \equiv \partial I/\partial x_1^{(1)} = 0 \tag{8}$$

(I thus being independent of $x_1^{(1)}$) and

$$L_i^{(1)} I \equiv \partial I/\partial x_i^{(1)} + \sum_{m=s+1}^{n} b_{im}^{(1)} \partial I/\partial x_m^{(1)} = 0 \qquad i = 2, \ldots, s. \tag{9}$$

The $b_{im}^{(1)}$ occurring in equation (9) are independent of $x_1^{(1)}$. Equation (9) is thus a complete Jacobian system of $n-1$ variables $x_i^{(1)}$, $i = 2, \ldots, n$. Proceeding in this way step by step we finally obtain $p = n - s$ functions I_α, $\alpha = 1, \ldots, p$, which simultaneously satisfy equation (5).

In practice we need not eliminate the dependent equations in (4). We work with all the n equations (4) and after the first step equate separately to zero the coefficients of different powers of $x_1^{(1)}$, thus possibly getting some extra equations. This procedure is followed at every step. At some intermediate step we obtain some identities like zero equal to zero, thus dropping the dependent equations. We thus have an algorithm for obtaining simultaneous integrals of a set of partial differential equations.

We have applied this method for obtaining the complete basis set of invariants for the groups of mathematical physics, described in § 1. It is true that, even for a single partial differential equation of n variables obtaining, the $n-1$ integrals may be a difficult proposition. Fortunately, in all the cases and in all the steps involved, we have been able, by a judicious choice of sequences in the solution of equation (7), to reduce the equations to the following types: $\mathrm{d}y + yf(x)\,\mathrm{d}x = 0$, $ax\,\mathrm{d}x + by\,\mathrm{d}y = 0$ and $by\,\mathrm{d}x + ax\,\mathrm{d}y = 0$. Only once in the case of SU(3) we are confronted with the differential equation $\mathrm{d}x/2z^2 = \mathrm{d}y/(24xz^3 - y^2) = -\mathrm{d}z/yz$, having the integrals $I_1 = 6x^2 - y/z$ and $I_2 = 2z - 4x^3 + xy/z$.

3. Examples

In this section we apply the procedure described in § 2 to the groups stated in § 1 and obtain the complete set of functionally independent bases of invariants for these groups.

3.1. Non-linear Klein–Gordon (KG) *equation*

In terms of the space variables $q_\alpha (\alpha = 1, 2, 3)$ and the reduced time variable $\tau = vt$, this equation is

$$\sum_\alpha \Psi_{\alpha\alpha} - \Psi_{\tau\tau} + V(\Psi) = 0 \tag{10}$$

where α and τ as subscripts mean partial derivatives with respect to q_α and τ, respectively.

3.1.1. General $V(\Psi)$. It has been shown (Rudra 1986a) that for a general form of non-linearity $V(\Psi)$, the symmetry group is the Poincaré group with generators

$$X^{\alpha} = -\mathrm{i}\partial/\partial q_{\alpha} \qquad X^{\tau} = -\mathrm{i}\partial/\partial\tau \qquad X^{\alpha}_{\mathrm{R}} = -\mathrm{i}\sum_{\beta\gamma} e_{\alpha\beta\gamma} q_{\beta}\partial/\partial q_{\gamma}$$
$$X^{\alpha}_{\mathrm{L}} = \tau\partial/\partial q_{\alpha} + q_{\alpha}\partial/\partial\tau \tag{11}$$

with structure constants

$$[X^{\alpha}, X^{\beta}_{\mathrm{R}}] = \mathrm{i}\sum_{\gamma} e_{\alpha\beta\gamma} X^{\gamma} \qquad [X^{\alpha}, X^{\beta}_{\mathrm{L}}] = \delta_{\alpha\beta} X^{\tau} \qquad [X^{\tau}, X^{\alpha}_{\mathrm{L}}] = X^{\alpha}$$
$$[X^{\alpha}_{\mathrm{R}}, X^{\beta}_{\mathrm{R}}] = \mathrm{i}\sum_{\gamma} e_{\alpha\beta\gamma} X^{\gamma}_{\mathrm{R}} \qquad [X^{\alpha}_{\mathrm{R}}, X^{\beta}_{\mathrm{L}}] = \mathrm{i}\sum_{\gamma} e_{\alpha\beta\gamma} X^{\gamma}_{\mathrm{L}} \tag{12}$$
$$[X^{\alpha}_{\mathrm{L}}, X^{\beta}_{\mathrm{L}}] = \mathrm{i}\sum_{\gamma} e_{\alpha\beta\gamma} X^{\gamma}_{\mathrm{R}}$$

where $e_{\alpha\beta\gamma}$ is the permutation symbol.

The complete set of functionally independent bases consists of the two well known invariants

$$I_1 = \sum_{\alpha} (X^{\alpha})^2 - (X^{\tau})^2$$
$$I_2 = S\Bigg\{ (X^{\tau})^2 \sum_{\alpha} (X^{\alpha}_{\mathrm{R}})^2 - \Big(\sum_{\alpha} X^{\alpha} X^{\alpha}_{\mathrm{R}}\Big)^2 + \Big(\sum_{\alpha} X^{\alpha} X^{\alpha}_{\mathrm{L}}\Big)^2$$
$$- \Big(\sum_{\alpha} (X^{\alpha})^2\Big)\Big(\sum_{\alpha} (X^{\alpha}_{\mathrm{L}})^2\Big) + 2\mathrm{i} X^{\tau} \sum_{\alpha\beta\gamma} e_{\alpha\beta\gamma} X^{\alpha} X^{\beta}_{\mathrm{R}} X^{\gamma}_{\mathrm{L}} \Bigg\}. \tag{13}$$

Here and below in similar positions S means that symmetrisation in the X has to be done. That these invariants form the complete basis for polynomial Casimir invariants is well known (Elliott and Dawber 1979). What we have obtained here is that they also form the complete basis of functionally independent invariants.

3.1.2. Linear KG equation, $V = V_0\Psi$. The symmetry group is a direct product of the Poincaré group and the Ψ-scaling operator $X_0 = \Psi\partial/\partial\Psi$, with X_0 commuting with all the generators of equation (11). The complete basis set in this case consists of three invariants, I_1, I_2 of equation (13) and $I_3 = X_0$.

3.1.3. $V = V_0\Psi^n$ ($n \neq 1, 3$) and $V = V_0 \exp(-c\Psi)$, complex $c \neq 0$. Both these cases have the Weyl group as the maximal symmetry group with an extra generator X_0 over and above those of the Poincaré group. For the case of the power-law non-linearity $V = V_0\Psi^n$ ($n \neq 1, 3$)

$$X_0 = \sum_{\alpha} q_{\alpha}\partial/\partial q_{\alpha} + \tau\partial/\partial\tau - [2\Psi/(n-1)]\partial/\partial\Psi \tag{14a}$$

and for an exponential-type non-linearity $V = V_0\exp(-c\Psi)$, $c \neq 0$,

$$X_0 = \sum_{\alpha} q_{\alpha}\partial/\partial q_{\alpha} + \tau\partial/\partial\tau + (2/c)\partial/\partial\Psi \tag{14b}$$

with extra commutator

$$[X^{\alpha}, X_0] = X^{\alpha} \qquad [X^{\tau}, X_0] = X^{\tau}. \tag{15}$$

The complete basis set here consists of the single invariant

$$I = I_2/I_1 \tag{16}$$

where I_1 and I_2 are given in equation (13).

3.1.4. $V=V_0\Psi^3$. The maximal symmetry group in this case is the conformal group, with the following five extra generators over and above those of the Poincaré group:

$$X_0=\sum_\alpha q_\alpha\partial/\partial q_\alpha+\tau\partial/\partial\tau-\Psi\partial/\partial\Psi$$
$$X^\alpha_{\rm A}=-({\rm i}/2)q^2X^\alpha+q_\alpha X_0\qquad X^\tau_{\rm A}=-({\rm i}/2)q^2X^\tau-\tau X_0 \tag{17a}$$

where $q^2=\Sigma_\alpha q_\alpha^2-\tau^2$, with the extra commutation relations

$$[X^\alpha,X_0]=X^\alpha\qquad[X^\alpha,X^\beta_{\rm A}]=-{\rm i}\delta_{\alpha\beta}X_0-\sum_\gamma e_{\alpha\beta\gamma}X^\gamma_{\rm R}\qquad[X^\alpha,X^\tau_{\rm A}]={\rm i}X^\alpha_{\rm L}$$
$$[X^\tau,X_0]=X^\tau\qquad[X^\tau,X^\alpha_{\rm A}]=-{\rm i}X^\alpha_{\rm L}\qquad[X^\tau,X^\tau_{\rm A}]={\rm i}X_0$$
$$[X^\alpha_{\rm R},X^\beta_{\rm A}]={\rm i}\sum_\gamma e_{\alpha\beta\gamma}X^\gamma_{\rm A}\qquad[X^\alpha_{\rm L},X^\beta_{\rm A}]=-\delta_{\alpha\beta}X^\tau_{\rm A}\qquad[X^\alpha_{\rm L},X^\tau_{\rm A}]=-X^\alpha_{\rm A} \tag{17b}$$
$$[X_0,X^\alpha_{\rm A}]=X^\alpha_{\rm A}\qquad[X_0,X^\tau_{\rm A}]=X^\tau_{\rm A}.$$

The complete basis set here consists of the following three invariants:

$$I_1=S\left\{X_0^2+\sum_\alpha[(X^\alpha_{\rm R})^2+(X^\alpha_{\rm L})^2]-2{\rm i}\left(\sum_\alpha X^\alpha X^\alpha_{\rm A}-X^\tau X^\tau_{\rm A}\right)\right\}$$

$$I_2=S\left\{X_0\sum_\alpha X^\alpha_{\rm R}X^\alpha_{\rm L}-{\rm i}X^\tau\sum_\alpha X^\alpha_{\rm R}X^\alpha_{\rm A}+{\rm i}X^\tau_{\rm A}\sum_\alpha X^\alpha X^\alpha_{\rm R}-\sum_{\alpha\beta\gamma}e_{\alpha\beta\gamma}X^\alpha X^\beta_{\rm L}X^\gamma_{\rm A}\right\}$$

$$\begin{aligned}I_3=S\Bigg\{&\left(\sum_\alpha X^\alpha X^\alpha_{\rm A}-X^\tau X^\tau_{\rm A}\right)^2-\left(\sum_\alpha(X^\alpha)^2-(X^\tau)^2\right)\left(\sum_\alpha(X^\alpha_{\rm A})^2-(X^\tau_{\rm A})^2\right)\\&-2{\rm i}X^\tau X^\tau_{\rm A}\sum_\alpha(X^\alpha_{\rm R})^2+2{\rm i}\left(\sum_\alpha(X^\alpha_{\rm L})^2\right)\left(\sum_\alpha X^\alpha X^\alpha_{\rm A}\right)\\&-2{\rm i}X_0X^\tau_{\rm A}\sum_\alpha X^\alpha X^\alpha_{\rm L}-2{\rm i}\left(\sum_\alpha X^\alpha_{\rm L}X^\alpha_{\rm A}\right)\left(\sum_\alpha X^\alpha X^\alpha_{\rm L}\right)+2{\rm i}X_0X^\tau\sum_\alpha X^\alpha_{\rm L}X^\alpha_{\rm A}\\&+2{\rm i}\left(\sum_\alpha X^\alpha_{\rm R}X^\alpha_{\rm A}\right)\left(\sum_\alpha X^\alpha X^\alpha_{\rm R}\right)-\left(\sum_\alpha X^\alpha_{\rm R}X^\alpha_{\rm L}\right)^2-X_0^2\sum_\alpha[(X^\alpha_{\rm R})^2+(X^\alpha_{\rm L})^2]\\&+2X^\tau\sum_{\alpha\beta\gamma}e_{\alpha\beta\gamma}X^\alpha_{\rm R}X^\beta_{\rm L}X^\gamma_{\rm A}+2X_0\sum_{\alpha\beta\gamma}e_{\alpha\beta\gamma}X^\alpha X^\beta_{\rm R}X^\gamma_{\rm A}\\&+2X^\tau_{\rm A}\sum_{\alpha\beta\gamma}e_{\alpha\beta\gamma}X^\alpha X^\beta_{\rm R}X^\gamma_{\rm L}\Bigg\}.\end{aligned} \tag{18}$$

3.1.5. Inhomogeneous KG *equation,* $V=V_0$. The symmetry group is the semi-direct product of the Ψ translation (with generators $X^\Psi=-{\rm i}\partial/\partial\Psi$) and the Weyl group (with generators given in § 3.1.3 for $n=0$). The extra commutation relation is

$$[X^\Psi,X_0]=2X^\Psi. \tag{19}$$

The complete basis set now consists of two invariants, X^Ψ/I_1 and I_2/I_1 where I_1 and I_2 are given in equation (13).

These are the complete set of potentials $V(\Psi)$ that can occur for a non-linear KG equation.

3.2. *Schrödinger equation for atoms and molecules*

The Coulombic system for the $k_{\rm e}$th electron of mass m and charge e at position $\boldsymbol{r}_{k_{\rm e}}$ ($k_{\rm e}=1,\ldots,N_{\rm e}$), and T types of nuclei with the $k_{\rm n}$th nucleus ($k_{\rm n}=1,\ldots,N_{\rm n}$) of type

n ($n = 1, \ldots, T$) having mass M_n and atomic number Z_n at the position $\boldsymbol{r}_{nk_n}$, described by the time-dependent Schrödinger equation, has a 14-parameter Lie group (Rudra 1986b) as the symmetry group. This Coulomb group has the generators

$$\begin{aligned}
&X^{0\alpha} = -\mathrm{i}\left(\sum_{nk_n} \partial/\partial r_{nk_n\alpha} + \sum_{k_e} \partial/\partial r_{k_e\alpha}\right) \qquad X^{t} = -\mathrm{i}\partial/\partial t \\
&X^{\Psi} = (M_0/\hbar)\Psi\partial/\partial\Psi \qquad X^{t\alpha} = tX^{0\alpha} + R_\alpha X^{\Psi} \\
&L^{\alpha} = -\mathrm{i}\sum_{\beta\gamma} e_{\alpha\beta\gamma}\left(\sum_{nk_n} r_{nk_n\beta}\partial/\partial r_{nk_n\gamma} + \sum_{k_e} r_{k_e\beta}\partial/\partial r_{k_e\gamma}\right) \\
&A^{\alpha} = -L^{\alpha} + 2\sum_{\beta\gamma} e_{\alpha\beta\gamma} R_\beta X^{0\gamma}
\end{aligned} \tag{20}$$

where

$$\boldsymbol{R} = \left(\sum_{nk_n} M_n \boldsymbol{r}_{nk_n} + m\sum_{k_e} \boldsymbol{r}_{k_e}\right)(M_0)^{-1}$$

and $M_0 = mN_e + \Sigma_n M_n N_n$.

Here α, β, γ in the subscripts of $\boldsymbol{r}$ and $\boldsymbol{R}$ denote the cartesian components. The non-vanishing commutation relations are

$$\begin{aligned}
&[X^{0\alpha}, X^{t\beta}] = -\mathrm{i}\delta_{\alpha\beta}X^{\Psi} \qquad [X^{0\alpha}, L^{\beta}] = \mathrm{i}\sum_{\gamma} e_{\alpha\beta\gamma}X^{0\gamma} \qquad [X^{0\alpha}, A^{\beta}] = \mathrm{i}\sum_{\gamma} e_{\alpha\beta\gamma}X^{0\gamma} \\
&[X^{t}, X^{t\alpha}] = -\mathrm{i}X^{0\alpha} \qquad [X^{t\alpha}, L^{\beta}] = \mathrm{i}\sum_{\gamma} e_{\alpha\beta\gamma}X^{t\gamma} \qquad [X^{t\alpha}, A^{\beta}] = \mathrm{i}\sum_{\gamma} e_{\alpha\beta\gamma}X^{t\gamma} \\
&[L^{\alpha}, L^{\beta}] = \mathrm{i}\sum_{\gamma} e_{\alpha\beta\gamma}L^{\gamma} \qquad [L^{\alpha}, A^{\beta}] = \mathrm{i}\sum_{\gamma} e_{\alpha\beta\gamma}A^{\gamma} \qquad [A^{\alpha}, A^{\beta}] = \mathrm{i}\sum_{\gamma} e_{\alpha\beta\gamma}L^{\gamma}.
\end{aligned} \tag{21}$$

The complete basis set in this cases has four invariants:

$$\begin{aligned}
&I_1 = \sum_{\alpha}(X^{0\alpha})^2 - 2X^{\Psi}X^{t} \qquad I_1 = X^{\Psi} \qquad I_3 = S\left\{\sum_{\alpha}(L^{\alpha} - A^{\alpha})^2\right\} \\
&I_4 = S\left\{\sum_{\alpha}\left[X^{\Psi}(L^{\alpha} + A^{\alpha}) + 2\sum_{\beta\gamma} e_{\alpha\beta\gamma}X^{0\beta}X^{t\gamma}\right]^2\right\}.
\end{aligned} \tag{22}$$

3.3. SU(3) of strong interaction physics

The strongly interacting hadrons are classified (Elliott and Dawber 1979) by different multiplets of SU(3), which has the generators

$$\begin{aligned}
&T_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad T_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad T_z = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&U_+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad U_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad V_+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\
&V_- = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix}.
\end{aligned} \tag{23}$$

The complete basis set here contains two invariants:

$$I_1 = T_z^2 + (T_+T_- + T_-T_+)/2 + (U_+U_- + U_-U_+)/2 + (V_+V_- + V_+V_-)/2 + \tfrac{3}{4}Y^2$$

$$I_2 = S\{Y(T_z^2 + T_+T_-) - Y^3/4 + (T_+U_+V_+ + T_-U_-V_-) - T_z(U_+U_- - V_+V_-) - Y(U_+U_- - V_+V_-)/2\}. \qquad (24)$$

Finally, it is again emphasised that these invariants form the complete set of functionally independent invariants.

Note added in proof. Recently we have reported work on maximal symmetry groups of the non-linear Klein-Gordon equation, the Hamilton-Jacobi equation for a relativistic particle in flat spacetime and quantum relativistic equations (Rudra 1986a, c). After publication of these papers we became aware of earlier work on similar topics by Fushchich and Shteler (1982, 1983) and Fushchich and Serov (1983).

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